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Green's function for a neutron in magnetically bound states

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Abstract. We construct an integral representation for the momentum space Green's function of a neutron in interaction with a straight current-carrying wire, which is valid for the negative-energy case. The energy eigenvalues and eigenfunctions for the neutron in magnetically bound states are obtained. We point out the connection with the positive-energy case that may provide the scattering amplitude.

1. Introduction

A neutron in the static magnetic field of a linear conductor carrying current in the z-direction has been investigated in order to construct its bound states. The problem is of interest with possible applications provided by this system in nuclear, atomic and solid state physics. The energy eigenvalues of the neutron in magnetically bound states are hydrogenic. The exact bound state spectrum and the accidental degeneracy due to a hidden $O(3)$ symmetry for an electrically neutral particle with magnetic moment in an external static magnetic field generated by a linear current were found almost 22 years ago [1]. The calculation was performed in momentum space. The spectrum and degeneracy are quite similar to those in the case of a three-dimensional non-relativistic Coulomb problem, although the interactions are quite different in the two cases [2]. Recently, a computation of the Coulomb wavefunctions in momentum space has been constructed for unbounded relativistic motion [3].

More recently, the system under consideration has been studied from different viewpoints. One decade ago it was demonstrated that neutrons can be confined classically in a static magnetic field [4]. One of the possible applications is the measurement of magnetic moments of electrically neutral particles. Nine years ago it was shown that neutrons bound in interaction with a straight current-carrying wire possess an infinite number of bound states and other applications have been pointed out [5].

In the space of bound states we have symmetry under the $SO(3)$ group, while for scattering states the symmetry group is $SO(2,1)$. The weak symmetry of the Schrödinger equation for a neutral particle with spin $\frac{1}{2}$ in a static magnetic field of linear current has been considered [6]. The energy eigenvalues and eigenfunctions have been obtained by two different methods. One method is based on the reduction of the Schrödinger equation in the coordinate representation

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to a fourth-order Hamburger equation [7]. The second method uses dynamical supersymmetry in quantum mechanics (SUSY QM) [8] which leads to the bound state spectrum and the corresponding eigenfunctions in the momentum representation [9]. SUSY QM has recently been reviewed in [10]. In [7] two interesting experiments to demonstrate the existence of magnetically bound neutron surface and bulk states have been discussed.

In this paper, we consider the Green's function in the momentum representation for the above problem. In this case we find a Green's function consisting of two parts: one non-singular part with no pole and the other part containing a singular term with poles. In essence we have derived an integral representation of such a Green's function, equation (65), that could be continued to positive energy states and then used in the determination of the scattering amplitude.

This work is organized in the following way. In section 2 we start by summarizing the essential features of the dynamical symmetries of the problem. In sections 3 and 4 we obtain an integral equation for the Green's function and the energy eigenvalues and eigenfunctions for the bound states in an elegant manner. As in the case of the Coulomb Green's function [2], an analytic continuation to the positive-energy case can lead to the form of the scattering amplitude. The difficulties in the complete determination of the scattering amplitude in this case are discussed in section 4 which contains the conclusions.

2. The negative-energy eigenvalues via $SO(3)$ dynamical symmetry

Consider an electrically neutral spin-$\frac{1}{2}$ particle of mass M and a magnetic moment $\mu\vec{\sigma}$ (a neutron) in interaction with an infinite straight wire carrying a current I and located along the z-axis. The magnetic field generated by the wire is given by (we use units with $c = \hbar = 1$)

$$\vec{B} = 2I\frac{(-y, x, 0)}{(x^2 + y^2)} \tag{1}$$

where x and y are Cartesian coordinates in the plane perpendicular to the wire. The Hamiltonian of the particle is given by

$$H = \frac{\vec{p}^2}{2M} + \mu\vec{\sigma}\cdot\vec{B} \tag{2}$$

where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices.

The motion along the z-axis is free and will be ignored in the following. Thus we obtain a two-dimensional problem with

$$H = \frac{\vec{p}^2}{2M} + 2I\mu\frac{(-y\sigma_1 + x\sigma_2)}{(x^2 + y^2)}. \tag{3}$$

If we consider the two cases with current parallel and antiparallel to the z-direction we have the two Hamiltonians

$$H_{\pm} = \frac{\vec{p}^2}{2M} \pm 2I\mu\frac{(-y\sigma_1 + x\sigma_2)}{(x^2 + y^2)} \tag{4}$$

where

$$H_+ = \sigma_3\, H_-\sigma_3 \tag{5}$$

so that $H_{\pm}$ have the same spectrum and the eigenstates are connected by the σ_3-matrix. Henceforth we consider H_+ and drop the subscript.

The problem formulated above has a dynamical symmetry [1]. From the form of the Hamiltonian there is one obvious constant of motion

$$J_3 = xp_y - yp_x + \tfrac{1}{2}\sigma_3. \tag{6}$$

Next using the Cartesian tensor notation we have the commutation rules for $i = 1, 2$

$$\frac{1}{\mathrm{i}}[p_i, H] = \frac{4I\mu}{r^4}\varepsilon_{jk}\sigma_j\left(x_k x_i - \tfrac{1}{2}\delta_{ik}r^2\right) \tag{7}$$

$$\frac{1}{\mathrm{i}}[x_i, H] = \frac{p_i}{M} \tag{8}$$

$$\frac{1}{\mathrm{i}}[\sigma_i, H] = \frac{4I\mu}{I^2}\sigma_3 x_i \tag{9}$$

where $\varepsilon_{ij} = -\varepsilon_{ji}$ and $\varepsilon_{12} = 1$ and repeated indices are summed.

Also, the 'Runge–Lenz vector' defined by

$$A_i = \frac{\varepsilon_{ij}x_j}{\sigma_1 x_2 - \sigma_2 x_1} + \frac{1}{4I\mu M}[p_i, J_3]_+ \qquad (i = 1, 2) \tag{10}$$

satisfies the following commutation rules:

$$[J_3, A_i] = \mathrm{i}\varepsilon_{ij}A_j \qquad [A_i, H] = 0 \qquad [A_i, A_j] = -\mathrm{i}HJ_3\frac{2M}{(2I\mu M)^2}\varepsilon_{ij}. \tag{11}$$

Thus for negative-energy $(-E)$ eigenstates of H we may put

$$J_i = A_i\left\{\frac{(2I\mu M)^2}{-2ME}\right\}^{1/2} \qquad (i = 1, 2) \tag{12}$$

we obtain

$$[J_\alpha, J_\beta] = \mathrm{i}\varepsilon_{\alpha\beta\gamma}J_\gamma \tag{13}$$

where $\alpha, \beta = 1, 2, 3$. Thus J_α generate the algebra of $SO(3)$ in the space of negative-energy eigenstates of H. For the space of positive-energy eigenstates of H we change $-E$ to $+E$ in the above and obtain the commutation rules of the $SO(2, 1)$ algebra.

The $SO(3)$ dynamical symmetry leads to an accidental degeneracy in the bound state spectrum. One can show that $(E < 0)$

$$\vec{A}^2 = 1 + \frac{2M}{(2I\mu M)^2}\left(J_3^2 + \tfrac{1}{4}\right) \tag{14}$$

and

$$H = -\frac{(2I\mu M)^2}{2M\left(\vec{J}^2 + \frac{1}{4}\right)}. \tag{15}$$

Thus we obtain the following bound state spectrum:

$$E_{j+\frac{1}{2}} = -\frac{(2I\mu M)^2}{2M\left(j + \frac{1}{2}\right)^2} \tag{16}$$

where $j = \frac{1}{2}, \frac{3}{2}, \ldots$ and $m = -j, -j+1, \ldots, j$; which exhibits the $(2j+1)$-fold degeneracy explicitly.

The calculation of the Green's function will be performed in the following, adapting Schwinger's calculation [2] to the present case, in the momentum representation leading to the bound state spectrum of equation (16) above.

3. Green's function for negative energies

It may be mentioned that even if the bound state spectrum is deduced on the basis of dynamical symmetry it is difficult to use it for the construction of wavefunctions in the coordinate representation which have an awkward form. As in the case of the Coulomb problem the momentum representation allows us to use the dynamical symmetry in an elegant manner. We construct the Green's function in momentum space following Schwinger's [2] technique for the Coulomb Green's function except for one important modification.

To go over to the momentum representation we put

$$\vec{r} = \mathrm{i}\vec{\nabla}_p \tag{17}$$

and use

$$\nabla_p^2 \frac{1}{2\pi} \ln|\vec{p} - \vec{p}'| = \delta(\vec{p} - \vec{p}') \tag{18}$$

so that

$$\frac{\mathrm{i}\vec{\nabla}_p}{\nabla_p^2}\Phi(\vec{p}) = \frac{\mathrm{i}}{2\pi}\int \Phi(\vec{p}')\frac{(\vec{p} - \vec{p}')}{|\vec{p} - \vec{p}'|^2}\,\mathrm{d}^2 p'. \tag{19}$$

The momentum space Green's function $G(\vec{p}, \vec{p}')$ for energy E satisfies

$$\left(E - \frac{\vec{p}^2}{2M}\right)G(\vec{p}, \vec{p}') \qquad - \frac{\mathrm{i}I\mu}{\pi}\int \mathrm{d}^2 p'' \frac{\sigma_i\,\varepsilon_{ij}(\vec{p} - \vec{p}'')_j}{|\vec{p} - \vec{p}''|^2} G(\vec{p}'', \vec{p}') = \delta(\vec{p} - \vec{p}') \tag{20}$$

where σ_i $(i = 1, 2, 3)$ are the Pauli matrices and $\varepsilon_{12} = 1 = -\varepsilon_{21}, \varepsilon_{ii} = 0$. In analogy with the construction of a momentum space as a stereographic projection of a three-dimensional sphere implemented in [2] we assume that

$$E = -\frac{p_0^2}{2M} \tag{21}$$

is real and negative $(p_0 > 0)$, we use the coordinates (n_0, n_i) where

$$n_0 = \frac{p_0^2 - \vec{p}^2}{p_0^2 + \vec{p}^2} \qquad n_i = \frac{2p_0 p_i}{p_0^2 + \vec{p}^2} \tag{22}$$

and

$$n_0^2 + \vec{n}^2 = 1. \tag{23}$$

It is convenient to treat $(n_0, \vec{n})$ as a three-dimensional vector n. Here and in the following three-dimensional vectors will be written without the vector symbol. The scalar product of two 3-vectors a and b will be written as $a \cdot b$, where $a = (a_0, a_1, a_2)$.

The element of area (solid angle) on the surface of the unit sphere of equation (20) is

$$\mathrm{d}\Omega = \left.\frac{\mathrm{d}n_1\,\mathrm{d}n_2}{n_0}\right|_{|n|=1}. \tag{24}$$

Now

$$\mathrm{d}n_1\,\mathrm{d}n_2 = \mathrm{d}^2 p\, J\left(\frac{p_i}{n_j}\right) \tag{25}$$

where the Jacobian $J(p_i/n_j)$ is given by

$$J\left(\frac{p_i}{n_j}\right) = \left(\frac{2p_0}{p_0^2 + \vec{p}^2}\right)^2 \det(\mathbb{1} - B) \tag{26}$$

where the matrix elements of B are

$$B_{ij} = \frac{2p_i\,p_j}{p_0^2 + \vec{p}^2}. \tag{27}$$

Since for a matrix N

$$\ln\det N = \operatorname{tr}\ln N \tag{28}$$

we obtain

$$\ln\det(\mathbb{1} - B) = \ln n_0 \tag{29}$$

where we use the property

$$B^2 = B(1 - n_0). \tag{30}$$

Thus

$$\mathrm{d}\Omega = \left(\frac{2p_0}{p_0^2 + \vec{p}^2}\right)^2 \mathrm{d}^2 p \tag{31}$$

which gives

$$\delta(\Omega - \Omega') = \left(\frac{p_0^2 + \vec{p}^2}{2p_0}\right)^2 \delta(\vec{p} - \vec{p}\,'). \tag{32}$$

Further,

$$\begin{aligned}(n - n')^2 &= (n_0 - n_0')^2 + (\vec{n} - \vec{n}\,')^2 \\ &= \frac{(1 + n_0)(1 + n_0')}{p_0^2}(\vec{p} - \vec{p}\,')^2.\end{aligned} \tag{33}$$

We define

$$\Gamma(\Omega, \Omega') = -\frac{p_0^4}{M(1 + n_0)^{3/2}(1 + n_0')^{3/2}} U^{-1}(n)\, G(\vec{p}, \vec{p}\,')\, U(n') \tag{34}$$

and

$$D(\Omega, \Omega') = -\frac{1}{4\pi} U^{-1}(n) \frac{2\mathrm{i}(\vec{\sigma} \wedge (\vec{p} - \vec{p}\,'))}{p_0} \frac{(1 + n_0)^{1/2}(1 + n_0')^{1/2}}{(n - n')^2}\, U(n') \tag{35}$$

so that we obtain the following integral equation for the modified Green's function:

$$\Gamma(\Omega, \Omega') - \nu \int D(\Omega, \Omega'')\, \Gamma(\Omega'', \Omega')\, \mathrm{d}\Omega'' = \delta(\Omega - \Omega') \tag{36}$$

where

$$\nu = \frac{2I\mu M}{p_0} \tag{37}$$

which *looks* covariant under three-dimensional rotations due to the notational artifice.

4. An integral representation for the Green's function

The introduction of the matrix $U(n)$ although permitted would appear to be redundant. However, if we omit it the form of $D(\Omega, \Omega')$ would not display the dependence on the 3-vectors n and n' in a convenient fashion, although the left-hand side of equation (31) should do so. We determine $U(n)$ by having $D(\Omega, \Omega')$ satisfy a differential equation where its rotational invariance is obvious.

We note that

$$\frac{\vec{\sigma}\wedge(\vec{p}-\vec{p}\,')}{(n-n')^2p_0^2}(1+n_0)(1+n_0') = \frac{\vec{\sigma}\wedge(\vec{p}-\vec{p}\,')}{|\vec{p}-\vec{p}\,'|^2}. \tag{38}$$

Also

$$\begin{aligned}(\vec{\sigma}\wedge\vec{\nabla}_p)\frac{(\vec{\sigma}\wedge(\vec{p}-\vec{p}\,')}{|\vec{p}-\vec{p}\,'|^2} &= 2\pi\,\delta(\vec{p}-\vec{p}\,')\\ &= 2\pi\left(\frac{(1+n_0)}{p_0}\right)^2\delta(\Omega-\Omega').\end{aligned} \tag{39}$$

Thus

$$(\vec{\sigma}\wedge\vec{\nabla}_p)(1+n_0)^{1/2}(1+n_0')^{1/2}U(n)\,D\,U^{-1}(n') = -\mathrm{i}p_0\left(\frac{1+n_0}{p_0}\right)^2\delta(\Omega-\Omega'). \tag{40}$$

Let

$$\hat{U}(n) = \sqrt{\tfrac{1}{2}(1+n_0)}\,U(n) \tag{41}$$

then

$$\hat{U}^{-1}(n)\frac{\mathrm{i}p_0}{1+n_0}(\vec{\sigma}\wedge\vec{\nabla}_p)\,\hat{U}(n)\,D(\Omega,\Omega') = \delta(\Omega-\Omega'). \tag{42}$$

The right-hand side of equation (42) suggests the use of polar coordinates, hence if the polar angles of $(n_0,\vec{n})$ are θ,ϕ;

$$\begin{aligned}p_1 &= p_0\tan\left(\tfrac{1}{2}\theta\right)\cos\phi,\\ p_2 &= p_0\tan\left(\tfrac{1}{2}\theta\right)\sin\phi.\end{aligned} \tag{43}$$

We then have

$$\frac{\mathrm{i}p_0}{1+n_0}\vec{\sigma}\wedge\vec{\nabla}_p = -\mathrm{i}\left(\sigma\cdot e_\phi\partial_\theta - \frac{\sigma\cdot e_p}{\sin\theta}\partial_\phi\right) \tag{44}$$

where e_p and e_ϕ are unit vectors.

Thus the operator acting on $D(\Omega,\Omega')$ in equation (42) involves at most first-order differential operators ∂_θ and ∂_ϕ. If it were invariant under rotations we can make $D(\Omega,\Omega')$ invariant too. The unique candidate in that case is the operator $a+b\sigma\cdot L$ where a and b are constants. We choose $\hat{U}(n)$ so that this holds true.

Since

$$\sigma\cdot L = -\mathrm{i}(\sigma\cdot e_\phi\partial_\theta + \sigma_3\partial_\phi - \cot(\theta) + \sigma\cdot e_p\,\partial_\phi) \tag{45}$$

the choice

$$\hat{U}(n) = f(\theta,\phi)\ \exp\left(\tfrac{1}{2}\mathrm{i}g(\theta,\phi)\sigma\cdot e_\phi\right) \tag{46}$$

maintains the $\mathrm{i}\sigma\cdot e_\phi\partial_\theta$ term (apart from other additional terms without derivatives).

Working out the derivatives it is easy to see that one must choose $g(\theta,\phi)=\theta$, which gives

$$\hat{U}^{-1}\left(-\mathrm{i}\vec{\sigma}\cdot\vec{e}_\phi\partial_\phi + \mathrm{i}\frac{\vec{\sigma}\cdot\vec{e}_p}{\sin\theta}\partial_\phi\right)\hat{U} = \sigma\cdot L - \mathrm{i}\hat{U}^{-1}\,\sigma\cdot e_\phi(\partial_\theta\hat{U}) + \mathrm{i}\hat{U}^{-1}\frac{\sigma\cdot e_p}{\sin\theta}(\partial_\phi\hat{U}) \tag{47}$$

where in the bracketed terms the derivatives act only on $\hat{U}$.

If we impose the condition that the last two terms are independent of the σ-matrices we obtain

$$\hat{U}(n) = \tfrac{1}{2}(1+n_0) + \tfrac{1}{2}\mathrm{i}\sin\theta\,\sigma\cdot e_\phi \tag{48}$$

and then

$$\hat{U}^{-1}(n)\frac{\mathrm{i}p_0}{1+n_0}\vec{\sigma}\wedge\vec{\nabla}_p\hat{U}(n) = \mathbb{1}+\sigma\cdot L. \tag{49}$$

Hence

$$(\mathbb{1}+\sigma\cdot L)\,D(\Omega,\Omega') = \delta(\Omega,\Omega'). \tag{50}$$

Our choice of $\hat{U}(n)$ gives

$$\hat{U}(n) = \sqrt{\frac{1+n_0}{2}}\left(1+\frac{\sigma_3\vec{\sigma}\cdot\vec{p}}{p_0}\right) \tag{51}$$

which allows us to calculate $D(\Omega,\Omega')$ explicitly by using equation (35). We obtain

$$D(\Omega,\Omega') = \frac{1}{2\pi}\frac{(1-\sigma\cdot n\,\sigma\cdot n')}{(n-n')^2}. \tag{52}$$

Although we know $D(\Omega,\Omega')$ in a closed form it is convenient to rewrite it in terms of the eigenfunctions of the operator $\mathbb{1}+\sigma\cdot L$. The eigenfunctions of the operators J^2, L^2, S^2 and J_3 where $S=\sigma/2$ and $J=L+S$ are known to be $Y^{\pm}_{jm}(\theta,\phi)$ where $\pm$ correspond to $j\pm\frac{1}{2}=\ell+\frac{1}{2}\mp\frac{1}{2}$, respectively, [11]. Also

$$(\mathbb{1}+\sigma\cdot L)\,Y^{\pm}_{jm}(\theta,\phi) = \pm\big(j+\tfrac{1}{2}\big)\,Y^{\pm}_{jm}(\theta,\phi) \tag{53}$$

so that

$$D(\Omega,\Omega') = \sum_{jm}\frac{1}{j+\frac{1}{2}}\big\{Y^{+}_{jm}(\theta,\phi)\,Y^{+\dagger}_{jm}(\theta',\phi') - Y^{-}_{jm}(\theta,\phi)\,Y^{-\dagger}_{jm}(\theta',\phi')\big\}. \tag{54}$$

Going back to equation (36) we obtain

$$\Gamma(\Omega,\Omega') = \sum_{jm}\left\{\frac{Y^{+}_{jm}(\theta,\phi)\,Y^{+\dagger}_{jm}(\theta',\phi')}{1-\nu/\big(j+\frac{1}{2}\big)} + \frac{Y^{-}_{jm}(\theta,\phi)\,Y^{-\dagger}_{jm}(\theta',\phi')}{1+\nu/\big(j+\frac{1}{2}\big)}\right\} \tag{55}$$

which for $\nu>0$ has poles at

$$\nu = \frac{2IM\mu}{p_0} = j+\tfrac{1}{2}. \tag{56}$$

This gives the bound state energies which are the same as before. The unnormalized bound state wavefunctions are $[1/(1+n_0)^{3/2}]\hat{U}^{-1}Y^{+}_{jm}(\theta,\phi)$.

Next, one can perform the sum indicated in equation (55) to write an integral representation for $\Gamma(\Omega,\Omega')$ which, in turn, leads to one for the Green's function $G(\vec{p},\vec{p}')$. This is done by using the identities

$$\frac{1}{1\mp\nu/\big(j+\frac{1}{2}\big)} = \frac{j+\frac{1}{2}}{j+\frac{1}{2}\mp\nu} = 1\pm\frac{\nu}{j+\frac{1}{2}\mp\nu} = 1\pm\nu/\big(j+\tfrac{1}{2}\big)+\frac{\nu^2}{\big(j+\frac{1}{2}\big)\big(j+\frac{1}{2}\mp\nu\big)} \tag{57}$$

and using

$$\frac{1}{n} = \int_0^1 \rho^{n-1}\,\mathrm{d}\rho \tag{58}$$

valid for $n>1$. Although each form of the identity leads to a different looking integral representation, they are related through integration by parts. We concentrate on the second form.

We have for $|\nu| < 1$,

$$\Gamma(\Omega, \Omega') = \delta(\Omega, \Omega') + \nu \int_0^1 (\rho^{-\nu}\Delta^+ - \rho^\nu \Delta^-)\, \mathrm{d}\rho \tag{59}$$

where

$$\Delta^\pm = \sum_{jm} \rho^{j-1/2}\, Y^\pm_{jm}(\theta, \phi)\, Y^{\pm\dagger}_{jm}(\theta', \phi'). \tag{60}$$

Using the explicit forms of $Y^\pm_{jm}(\theta, \phi)$ one can show that

$$\begin{aligned} \Delta^+ &= \frac{1}{4\pi}(\partial_\rho \rho + \sigma \cdot L)\Phi \\ \Delta^- &= \frac{1}{4\pi}\left(-\frac{1}{\rho}\sigma \cdot L + \partial_\rho\right)\Phi \end{aligned} \tag{61}$$

where

$$\Phi = \left((1-\rho)^2 + \rho(n-n')^2\right)^{-1/2}. \tag{62}$$

Calculating the derivatives we obtain

$$\Delta^+ = \frac{1}{4\pi}(1 - \rho\,\sigma \cdot n\,\sigma \cdot n')\Phi^3 \qquad \Delta^- = \frac{1}{4\pi}(\rho - \sigma \cdot n\,\sigma \cdot n')\Phi^3. \tag{63}$$

Thus

$$\Gamma(\Omega, \Omega') = \delta(\Omega - \Omega') + \frac{\nu}{4\pi}\int_0^1 \mathrm{d}\rho \left(\rho^{-\nu}(1 - \rho\sigma \cdot n\,\sigma \cdot n') - \rho^\nu(\sigma \cdot n\,\sigma \cdot n' - \rho)\right)\Phi^3. \tag{64}$$

Substituting this in equation (34) we obtain the integral representation for $G(\vec{p}, \vec{p}\,')$:

$$\begin{aligned} G(\vec{p}, \vec{p}\,') = {} & \frac{\delta(\vec{p} - \vec{p}\,')}{E - T} + \frac{\nu}{8\pi M}\frac{1}{E - T}\frac{1}{E - T'}\frac{1}{|\vec{p} - \vec{p}\,'|^2} \\ & \times\left\{\left(2M\left(E - \tfrac{1}{2}(T + T')\right) - \mathrm{i}\sigma_3(\vec{p} \wedge \vec{p}\,') + \tfrac{1}{2}|\vec{p} - \vec{p}\,'|^2\right)\right. \\ & \times(I(-\nu) - I(\nu) + I(1+\nu) - I(1-\nu)) \\ & \left. + \mathrm{i}p_0\,\vec{\sigma} \wedge (\vec{p} - \vec{p}\,')(I(\nu) + I(-\nu) + I(1+\nu) + I(1-\nu))\right\} \end{aligned} \tag{65}$$

where p_0 and ν are given by equations (21) and (37). Also,

$$I(\nu) = \int_0^1 \mathrm{d}\rho \frac{\rho^\nu (n - n')^2}{\left\{(1-\rho)^2 + \rho(n-n')^2\right\}^{3/2}} \tag{66}$$

$$T = \frac{p^2}{2M} \qquad T' = \frac{p'^2}{2M} \tag{67}$$

and

$$(n - n')^2 = -\frac{2E}{M}\frac{|\vec{p} - \vec{p}\,'|^2}{(E - T)(E - T')}. \tag{68}$$

This integral representation is valid for $|\nu| < 1$. This restriction may be removed by redefining the integral as a contour integral.

5. Discussions and conclusion

First we presented a review of the calculation of the negative-energy eigenvalues via $SO(3)$ dynamical symmetry associated with a neutron in magnetically bound states. Such an $SO(3)$ symmetry is quite analogous to that of the $O(4)$ symmetry related to an accidental degeneracy of the tridimensional non-relativistic Coulomb problem [6]. Next we obtained an integral representation for the Green's function for a neutron in interaction with a linear current. Such an integral representation emerges from an expansion of the Green's function for which the poles are the energy eigenvalues and the residues are (exactly) the energy eigenfunctions of the problem modified by a unitary transformation so that $1/\left[(1+n_0)^{3/2}\right]\hat{U}^{-1}Y^{+}_{jm}(\theta,\phi)$ are the unnormalized bound state wavefunctions of the interaction between the neutron magnetic dipole and the circular magnetic field of a linear conductor with current. In this case we found a Green's function consisting of two parts: one non-singular term with no pole and another term containing a singular term with poles. The results were deduced for the negative-energy case. We have considered the current parallel to the z-direction so that the Hamiltonian in this case is $H_+ = H(2\mu I)$ which is given by equation (3) in the explicit form. Let $H_- = H(-2\mu I)$ be the Hamiltonian for the case with current antiparallel to the z-direction. Then both are related by means of a unitary transformation so that $H_\pm$ have the same energy spectrum. The Green's function given by equation (55) constitutes a new scenario of Schrödinger–Green's function because it contains an additional regular part. The main reason for this behaviour of the neutron magnetically bound Green's function is due to the fact that neither Y^{+}_{jm} nor Y^{-}_{jm} can be used to make a complete set, but only considering both spinors $Y^{\pm}_{jm}$ is it possible to construct the following relation of completeness:

$$\delta(\Omega,\Omega') = \sum_{jm}\left\{Y^{+}_{jm}(\theta,\phi)\,Y^{+\dagger}_{jm}(\theta',\phi') + Y^{-}_{jm}(\theta,\phi)\,Y^{-\dagger}_{jm}(\theta',\phi')\right\}.$$

The energy eigenvalues and eigenfunctions constructed are in accord with those obtained by the method in the coordinate representation, which is based on the transformation of the Schrödinger equation in a fourth-order Hamburger equation [7] and another method in the momentum representation in which this system becomes a Pöschl–Teller potential with a dynamical supersymmetry [9]. In all these works only the negative-energy case has been considered. However, the scattering of neutrons for unbound states associated with the continuous positive spectrum can readily be implemented from the integral representation for the Green's function, equation (65), deduced here so that one can construct the scattering amplitude in an analogous method developed for the Coulomb scattering problem in [2]. This occurs because equation (55) is valid not only for the negative-energy eigenvalues but is also satisfied by the positive-energy continuum. By analytic continuation to $E > 0$ one can find the Green's function for scattering states so that we can construct the scattering amplitude if the distortion-free propagators can be identified. In contrast to the Coulomb problem it is not easy to solve the scattering problem in the coordinate representation by separation of variables. Although a general momentum transfer dependence of $(\vec{p}-\vec{p}\,')^{-2}$ is indicated, many details need to be worked out. This will be done elsewhere.

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